Admissions Testing Service

STEP Mark Scheme 2015

Mathematics

STEP 9465/9470/9475

October 2015

Test

1. (i)

$$I_n - I_{n+1} = \int_0^\infty \frac{1}{(1+u^2)^n}du - \int_0^\infty \frac{1}{(1+u^2)^{n+1}}du = \int_0^\infty \frac{1+u^2-1}{(1+u^2)^{n+1}}du$$

$$= \int_0^\infty \frac{u^2}{(1+u^2)^{n+1}}du$$

B1

$$= \int_0^\infty u\frac{u}{(1+u^2)^{n+1}}du = \left[u\frac{-1}{2n(1+u^2)^n}\right]_0^\infty - \int_0^\infty \frac{-1}{2n(1+u^2)^n}du$$

integrating by parts **M1 A1**

$$= 0 + \frac{1}{2n}\int_0^\infty \frac{1}{(1+u^2)^n}du = \frac{1}{2n}I_n$$

A1* **(4)**

$I_{n+1} = I_n - \frac{1}{2n}I_n = \frac{2n-1}{2n}I_n$ **M1**

$= \frac{(2n-1)(2n-3)\ldots(1)}{(2n)(2n-2)\ldots(2)}I_1$ **M1**

$I_1 = \int_0^\infty \frac{1}{(1+u^2)}du = [\tan^{-1}u]_0^\infty = \frac{\pi}{2}$ **B1**

$\frac{(2n-1)(2n-3)\ldots(1)}{(2n)(2n-2)\ldots(2)} = \frac{(2n)(2n-1)(2n-2)(2n-3)\ldots(2)(1)}{[(2n)(2n-2)\ldots(2)]^2} = \frac{(2n)!}{[2^n n!]^2} = \frac{(2n)!}{2^{2n}(n!)^2}$ **M1**

Thus $I_{n+1} = \frac{(2n)!}{2^{2n}(n!)^2}\frac{\pi}{2} = \frac{(2n)!\pi}{2^{2n+1}(n!)^2}$ **A1*** **(5)**

(ii)

$$J = \int_0^\infty f((x-x^{-1})^2)dx = \int_\infty^0 f((u^{-1}-u)^2).-u^{-2}du = \int_0^\infty x^{-2}f((x-x^{-1})^2)dx$$

using the substitution $u = x^{-1}$, $\frac{du}{dx} = -x^{-2}$ and then the substitution $u = x$, $\frac{du}{dx} = 1$ **M1A1***

$$2J = \int_0^\infty f((x-x^{-1})^2)dx + \int_0^\infty x^{-2}f((x-x^{-1})^2)dx = \int_0^\infty f((x-x^{-1})^2)(1+x^{-2})dx$$

So $J = \frac{1}{2}\int_0^\infty f((x-x^{-1})^2)(1+x^{-2})dx$ **M1A1***

Using the substitution $u = x - x^{-1}$, $\frac{du}{dx} = 1 + x^{-2}$,

$$J=\frac{1}{2}\int_0^\infty f((x-x^{-1})^2)(1+x^{-2})dx=\frac{1}{2}\int_{-\infty}^\infty f(u^2)\,du=\int_0^\infty f(u^2)du$$ M1A1* (6)

(iii)

$$\int_0^\infty \frac{x^{2n-2}}{(x^4-x^2+1)^n}dx=\int_0^\infty \frac{x^{-2}}{(x^2-1+x^{-2})^n}dx=\int_0^\infty \frac{x^{-2}}{((x-x^{-1})^2+1)^n}dx$$ **M1A1**

$$=\int_0^\infty \frac{1}{(u^2+1)^n}du$$ **M1**

So

$$\int_0^\infty \frac{x^{2n-2}}{(x^4-x^2+1)^n}dx=\int_0^\infty \frac{1}{(u^2+1)^n}du=I_n$$ **M1**

$$=\frac{\big(2(n-1)\big)!\pi}{2^{2(n-1)+1}\big((n-1)!\big)^2}=\frac{(2n-2)!\pi}{2^{2n-1}\big((n-1)!\big)^2}$$ **A1 (5)**

2. (i) True. **B1**

$m = 1000$ **B1**

If $n \geq 1000$, then $1000 \leq n$, so $1000n \leq n^2$, i.e. $(1000n) \leq (n^2)$ **M1A1 (4)**

(ii) False. **B1**

E. G. Let $s_n = 1$ and $t_n = 2$ for n odd, and $s_n = 2$ and $t_n = 1$ for n even. **B1**

Then $\nexists m$ for which for $n \geq m$, $s_n \leq t_n$, nor $t_n \leq s_n$ **M1**

So it is not the case that $(s_n) \leq (t_n)$, but nor is it the case that $(t_n) \leq (s_n)$ **A1 (4)**

(iii) True. **B1**

$(s_n) \leq (t_n)$ means that there exists a positive integer, say m_1, for which for $n \geq m_1$, $s_n \leq t_n$. **E1**

$(t_n) \leq (u_n)$ means that there exists a positive integer, say m_2, for which for $n \geq m_2$, $t_n \leq u_n$. **E1**

Then if $m = \max(m_1, m_2)$, **B1**

for $n \geq m$, $s_n \leq t_n \leq u_n$, and so $(s_n) \leq (u_n)$ **A1 (5)**

(iv) True. **B1**

$m = 4$ **B1**

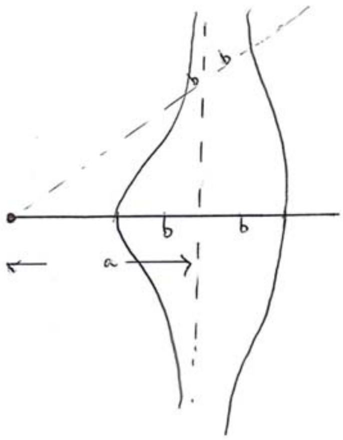
Assume $k^2 \leq 2^k$ for some value $k \geq 4$. **B1**

Then $(k+1)^2 = \left(\frac{k+1}{k}\right)^2 k^2 = \left(1+\frac{1}{k}\right)^2 k^2 \leq \left(1+\frac{1}{4}\right)^2 k^2 = \frac{25}{16}k^2 < 2k^2 \leq 2 \times 2^k = 2^{k+1}$ **M1A1**

$4^2 = 2^4$ **B1**

so by the principle of mathematical induction, $n^2 \leq 2^n$ for $n \geq 4$, and thus $(n^2) \leq (2^n)$ **A1 (7)**

3. (i)



Symmetry about initial line **G1**
Two branches **G1**
Shape and labelling **G1 (3)**

If $|r - a\sec\theta| = b$, then $r - a\sec\theta = b$ or $r - a\sec\theta = -b$

So $r = a\sec\theta + b$ or $r = a\sec\theta - b$ **M1A1**

If $\sec\theta < 0$, $a\sec\theta + b < -a + b < 0$ as $a > b$ and $a\sec\theta - b < -a - b < 0$ as a and b are both positive, and thus in both cases, $r < 0$ which is not permitted. **B1**

If $\sec\theta > 0$, $a\sec\theta + b > a + b > 0$ and $a\sec\theta - b > a - b > 0$ giving $r > 0$

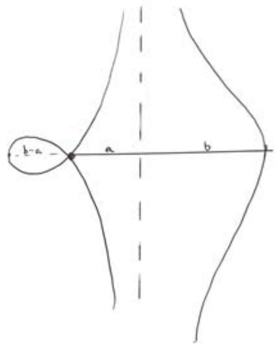
so $\sec\theta > 0$ as required. **B1** **(4)**

So $r = a\sec\theta \pm b$, thus points satisfying (*) lie on a certain conchoid of Nicomedes with *A* being the pole (origin), **B1**

d being *b*, **B1**

and L being the line $r = a\sec\theta$. **B1 (3)**

(ii)



Symmetry about initial line **G1**
Two branches **G1**
Loop, shape and labelling **G1**

If $a < b$, then the curve has two branches, $r = a\sec\theta + b$ with $\sec\theta > 0$ and $r = a\sec\theta + b$ with $\sec\theta < 0$, the endpoints of the loop corresponding to $\sec\theta = \frac{-b}{a}$. **B1 (4)**

In the case $a = 1$ and $b = 2$, $\sec\theta = \frac{-2}{1} = -2$ so $\theta = \pm\frac{2\pi}{3}$

Area of loop

$= 2 \times \frac{1}{2}\int_{\frac{2\pi}{3}}^{\pi}(\sec\theta + 2)^2 d\theta$ **M1A1**

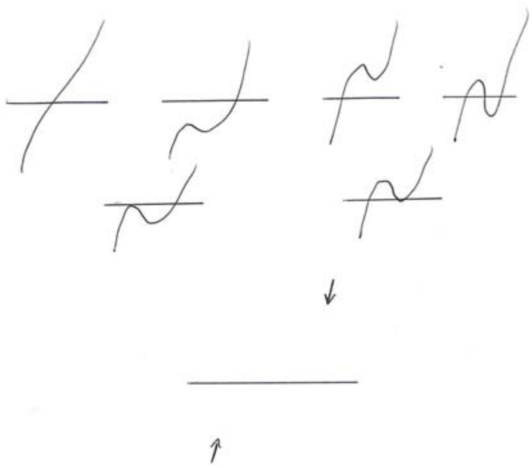
$= \int_{\frac{2\pi}{3}}^{\pi}\sec^2\theta + 4\sec\theta + 4\,d\theta = [\tan\theta + 4\ln|\sec\theta + \tan\theta| + 4\theta]_{\frac{2\pi}{3}}^{\pi}$ **M1A1**

$= 4\pi - \left(-\sqrt{3} + 4\ln\left|-2 - \sqrt{3}\right| + \frac{8\pi}{3}\right) = \frac{4\pi}{3} + \sqrt{3} - 4\ln\left|2 + \sqrt{3}\right|$ **M1A1 (6)**

4. (i) $y = z^3 + az^2 + bz + c$ is continuous.

For $z \to -\infty$, $y \to -\infty$ and for $z \to \infty$, $y \to \infty$. **B1**

So the sketch of this graph must be one of the following:-



B1

Hence, it must intersect the z axis at least once, and so there is at least one real root of

$z^3 + az^2 + bz + c = 0$ **B1 (3)**

(ii) $z^3 + az^2 + bz + c = (z - z_1)(z - z_2)(z - z_3)$ **M1**

Thus $a = (-z_1 - z_2 - z_3) = -S_1$ **A1**

$b = (z_2 z_3 + z_3 z_1 + z_1 z_2) = \frac{(z_1+z_2+z_3)^2 - ({z_1}^2+{z_2}^2+{z_3}^2)}{2} = \frac{{S_1}^2 - S_2}{2}$ **A1**

and, as ${z_1}^3 + a{z_1}^2 + bz_1 + c = 0$, ${z_2}^3 + a{z_2}^2 + bz_2 + c = 0$, ${z_3}^3 + a{z_3}^2 + bz_3 + c = 0$

adding these three equations we have,

$({z_1}^3 + {z_2}^3 + {z_3}^3) + a({z_1}^2 + {z_2}^2 + {z_3}^2) + b(z_1 + z_2 + z_3) + 3c = 0$ **M1**

(Alternatively,

$(z_1 + z_2 + z_3)^3 =$

$$({z_1}^3 + {z_2}^3 + {z_3}^3) + 3({z_1}^2 z_2 + {z_2}^2 z_3 + {z_3}^2 z_1 + {z_1}^2 z_3 + {z_2}^2 z_1 + {z_3}^2 z_2) + 6z_1 z_2 z_3$$

$({z_1}^2 + {z_2}^2 + {z_3}^2)(z_1 + z_2 + z_3) = ({z_1}^3 + {z_2}^3 + {z_3}^3) + ({z_1}^2 z_2 + {z_2}^2 z_3 + {z_3}^2 z_1 + {z_1}^2 z_3 + {z_2}^2 z_1 + {z_3}^2 z_2)$)

So $S_3 - S_1 S_2 + \frac{{S_1}^2 - S_2}{2} S_1 + 3c = 0$ **M1**

Thus $6c = \left(3S_1 S_2 - {S_1}^3 - 2S_3\right)$ **A1* (6)**

(iii) Let $z_k = r_k(\cos\theta_k + i\sin\theta_k)$ for $k = 1, 2, 3$ **M1**

Then $z_k{}^2 = r_k{}^2(\cos 2\theta_k + i\sin 2\theta_k)$ and $z_k{}^3 = r_k{}^3(\cos 3\theta_k + i\sin 3\theta_k)$ by de Moivre **M1**

As

$$\sum_{k=1}^{3} r_k \sin\theta_k = 0$$

$$\sum_{k=1}^{3} r_k{}^2 \sin 2\theta_k = 0$$

$$\sum_{k=1}^{3} r_k{}^3 \sin 3\theta_k = 0$$

$$Im\left(\sum_{k=1}^{3} z_k\right) = 0$$

$$Im\left(\sum_{k=1}^{3} z_k{}^2\right) = 0$$

$$Im\left(\sum_{k=1}^{3} z_k{}^3\right) = 0$$

and so S_1, S_2, and S_3 are real, **M1**

and therefore so are a, b, and c **A1**

Hence, as z_1, z_2, and z_3 are the roots of $z^3 + az^2 + bz + c = 0$ with a, b, and c real, by part (i), at least one of z_1, z_2, and z_3 is real. **M1**

So for at least one value of k, $r_k(\cos\theta_k + i\sin\theta_k)$ is real and thus, $\sin\theta_k = 0$,

and as $-\pi < \theta_k < \pi$, $\theta_k = 0$ as required. **A1 (6)**

If $\theta_1 = 0$ then z_1 is real. z_2 and z_3 are the roots of $(z - z_2)(z - z_3) = 0$

which is $z^2 + (-z_2 - z_3)z + z_2 z_3 = 0$ (say $z^2 + pz + q = 0$)

$p = -z_2 - z_3 = a + z_1$ and $q = z_2 z_3 = -\frac{c}{z_1}$ and so the quadratic of which z_2 and z_3 are the roots has real coefficients. Thus $z_2,\ z_3 = \frac{-p \pm \sqrt{p^2 - 4q}}{2}$. ($z_1 \neq 0$ because $r_k > 0$) **B1**

If $p^2 - 4q < 0$, **M1**

Thus $\cos\theta_2 = \cos\theta_3$, and so $\theta_2 = \pm\theta_3$, as $-\pi < \theta_k < \pi$.

But $\sin\theta_2 = -\sin\theta_3$ and so $\theta_2 = -\theta_3$. **M1 A1**

If $p^2 - 4q \geq 0$, then z_2 and z_3 are real roots, so $\sin\theta_2 = \sin\theta_3 = 0$, and thus $\theta_2 = \theta_3 = 0$, so $\theta_2 = -\theta_3$. **B1 (5)**

5. (i) Having assumed that $\sqrt{2}$ is rational (step 1), $\sqrt{2} = {}^{p}/_{q}$, where $p, q \in \mathbb{Z}, q \neq 0$ **B1**

Thus from the definition of S (step 2), as $q \in \mathbb{Z}$ and $\sqrt{2} = q \times {}^{p}/_{q} = p \in \mathbb{Z}$, so $q \in S$ proving step 3. **B1 (2)**

If $k \in S$, then k is an integer and $k\sqrt{2}$ is an integer. **B1**

So $(\sqrt{2} - 1)k = k\sqrt{2} - k$ is an integer, **B1**

and $(\sqrt{2} - 1)k\sqrt{2} = 2k - k\sqrt{2}$ which is an integer and so $(\sqrt{2} - 1)k \in S$ proving step 5. **B1 (3)**

$1 < \sqrt{2} < 2$ and so **M1**

$0 < \sqrt{2} - 1 < 1$, and thus $0 < (\sqrt{2} - 1)k < k$ **A1**

and thus this contradicts step 4 that k is the smallest positive integer in S as $(\sqrt{2} - 1)k$ has been shown to be a smaller positive integer and is in S . **A1 (3)**

(ii) If $2^{2/3}$ is rational, then $2^{2/3} = {}^{p}/_{q}$, where $p, q \in \mathbb{Z}, q \neq 0$

So $\left(2^{2/3}\right)^2 = \left({}^{p}/_{q}\right)^2$, that is $2^{4/3} = {}^{p^2}/_{q^2}$, which can be written $2 \times 2^{1/3} = {}^{p^2}/_{q^2}$ **M1**

and hence $2^{1/3} = {}^{p^2}/_{2q^2}$ proving that $2^{1/3}$ is rational. **A1**

If $2^{1/3}$ is rational, then $2^{1/3} = {}^{p}/_{q}$, where $p, q \in \mathbb{Z}, q \neq 0$ **M1**

and so $2^{2/3} = {}^{p^2}/_{q^2}$ proving that $2^{2/3}$ is rational and that $2^{1/3}$ is rational only if $2^{2/3}$ is rational. **A1 (4)**

Assume that $2^{1/3}$ is rational.

Define the set T to be the set of positive integers with the following property: n is in T if and only if $n2^{1/3}$ and $n2^{2/3}$ are integers. **B1**

The set T contains at least one positive integer as if $2^{1/3} = {}^{p}/_{q}$, where $p, q \in \mathbb{Z}, q \neq 0$, then $q^2 2^{1/3} = q^2 \times {}^{p}/_{q} = pq \in \mathbb{Z}$ and $q^2 2^{2/3} = q^2 \times {}^{p^2}/_{q^2} = p^2 \in \mathbb{Z}$, so $q^2 \in T$. **M1A1**

Define t to be the smallest positive integer in T. Then $t2^{1/3}$ and $t2^{2/3}$ are integers. **B1**

Consider $t\left(2^{2/3} - 1\right)$. $t\left(2^{2/3} - 1\right) = t2^{2/3} - t$ which is the difference of two integers and so is itself an integer. $t\left(2^{2/3} - 1\right) \times 2^{1/3} = 2t - t2^{1/3}$ which is an integer,

and $t\left(2^{2/3} - 1\right) \times 2^{2/3} = 2^{4/3}t - t2^{2/3} = 2 \times 2^{1/3}t - t2^{2/3}$ which is an integer.

Thus $t\left(2^{2/3}-1\right)$ is in T. **M1A1**

$1<2^{2/3}<2$ and so $0<2^{2/3}-1<1$, and thus $0<t\left(2^{2/3}-1\right)<t$, and thus this contradicts that t is the smallest positive integer in T as $t\left(2^{2/3}-1\right)$ has been shown to be a smaller positive integer and is in T. **M1A1 (8)**

6. (i) $w, z \in \mathbb{R} \Longrightarrow u, v \in \mathbb{R}$ **B1**

For $w, z \in \mathbb{R}$, we require to solve $w + z = u$, $w^2 + z^2 = v$ **M1**

$$w^2 + (u - w)^2 = v$$

$$2w^2 - 2uw + (u^2 - v) = 0$$

$$w = \frac{2u \pm \sqrt{4u^2 - 8u^2 + 8v}}{4} = \frac{u \pm \sqrt{2v - u^2}}{2}$$

$$z = \frac{u \mp \sqrt{2v - u^2}}{2}$$

M1A1

So for , $z \in \mathbb{R}$, as $u = w + z$ must be real, $v = w^2 + z^2$ must be real, and $2v - u^2 \geq 0$

i.e. $u^2 \leq 2v$ **B1* (5)**

(ii) $u = w + z \Longrightarrow u^2 = w^2 + z^2 + 2wz$ so if $w^2 + z^2 - u^2 = -\frac{2}{3}$, then $-2wz = -\frac{2}{3}$

so $3wz = 1$ **M1A1**

$$w^3 + z^3 = (w + z)(w^2 + z^2 - wz) = u(u^2 - 3wz) = u(u^2 - 1)$$

M1A1

Thus if $w^3 + z^3 - \lambda u = -\lambda$, $u(u^2 - 1) = \lambda(u - 1)$ **M1A1**

Thus $(u - 1)(u(u + 1) - \lambda) = 0$, **M1**

$(u - 1)(u^2 + u - \lambda) = 0$ **M1A1**

Thus $u = 1$ or $u = \frac{-1 \pm \sqrt{1 + 4\lambda}}{2}$

So as $\lambda \in \mathbb{R}$ and $\lambda > 0$, the values of u are real. **B1**

There are three distinct values of u unless $\frac{-1 \pm \sqrt{1 + 4\lambda}}{2} = 1$ in which case $\pm\sqrt{1 + 4\lambda} = 3$, i.e. $\lambda = 2$

M1A1 (12)

For $w, z \in \mathbb{R}$, from (i) we require $u \in \mathbb{R}$ which it is, $u^2 - \frac{2}{3} \in \mathbb{R}$ which it is, and $u^2 \leq 2\left(u^2 - \frac{2}{3}\right)$ in other words $u^2 \geq \frac{4}{3}$. **M1**

So w and z need not be real. A counterexample would be $u = 1$ **B1**

for then $w + z = 1$, $w^2 + z^2 = \frac{1}{3}$, so $w^2 + (1 - w)^2 = \frac{1}{3}$, i.e. $2w^2 - 2w + \frac{2}{3} = 0$ in which case the discriminant is $-\frac{4}{3} < 0$ so $w \notin \mathbb{R}$. **B1 (3)**

7. $D^2x^a = D(D(x^a)) = D\left(x\frac{d}{dx}(x^a)\right) = D(xax^{a-1})$ **M1**

$= D(ax^a) = x\frac{d}{dx}(ax^a) = xa^2x^{a-1} = a^2x^a$ **M1A1 (3)**

(i) Suppose $D^kP(x)$ is a polynomial of degree r i.e. $D^kP(x) = a_rx^r + a_{r-1}x^{r-1} + \cdots + a_0$ for some integer k. **B1**

Then $D^{k+1}P(x) = D(a_rx^r + a_{r-1}x^{r-1} + \cdots + a_0) = x\frac{d}{dx}(a_rx^r + a_{r-1}x^{r-1} + \cdots + a_0)$

$= x(ra_rx^{r-1} + (r-1)a_{r-1}x^{r-2} + \cdots + a_1) = ra_rx^r + (r-1)a_{r-1}x^{r-1} + \cdots + a_1x$

which is a polynomial of degree r. **M1A1**

Suppose $P(x) = b_rx^r + b_{r-1}x^{r-1} + \cdots + b_0$, then

$DP(x) = x\frac{d}{dx}(b_rx^r + b_{r-1}x^{r-1} + \cdots + b_0) = rb_rx^r + (r-1)b_{r-1}x^{r-1} + \cdots + b_1x$ so the result is true for $n = 1$, **M1A1**

and we have shown that if it is true for $n = k$, it is true for $n = k + 1$. Hence by induction, it is true for any positive integer. **B1 (6)**

(ii) Suppose $D^k(1-x)^m$ is divisible by $(1-x)^{m-k}$ i.e. $D^k(1-x)^m = f(x)(1-x)^{m-k}$ for some integer k, with $k < m - 1$. **B1**

Then $D^{k+1}(1-x)^m = D(f(x)(1-x)^{m-k}) = x\frac{d}{dx}(f(x)(1-x)^{m-k})$

$= x(f'^{(x)}(1-x)^{m-k} - (m-k)f(x)(1-x)^{m-k-1})$

$= x(1-x)^{m-k-1}\left(f'^{(x)}(1-x) - (m-k)f(x)\right)$ which is divisible by $(1-x)^{m-(k+1)}$. **M1A1**

$D(1-x)^m = x\frac{d}{dx}((1-x)^m) = -mx(1-x)^{m-1}$ so result is true for $n = 1$. **M1A1**

We have shown that if it is true for $n = k$, it is true for $n = k + 1$. Hence by induction, it is true for any positive integer $< m$. **B1 (6)**

(iii)

$(1-x)^m = \sum_{r=0}^{m}\binom{m}{r}(-x)^r = \sum_{r=0}^{m}(-1)^r\binom{m}{r}x^r$ **M1**

So

$D^n(1-x)^m = \sum_{r=0}^{m}(-1)^r\binom{m}{r}D^nx^r = \sum_{r=0}^{m}(-1)^r\binom{m}{r}r^nx^r$ **M1A1**

But by (ii), $D^n(1-x)^m$ is divisible by $(1-x)^{m-n}$ and so $D^n(1-x)^m = g(x)(1-x)^{m-n}$, and thus if $x = 1$, $D^n(1-x)^m = 0$, and hence

$\sum_{r=0}^{m}(-1)^r\binom{m}{r}r^n = 0$ **M1A1* (5)**

8. (i) $x = r\cos\theta \implies \frac{dx}{d\theta} = -r\sin\theta + \frac{dr}{d\theta}\cos\theta$ **M1A1**

and $y = r\sin\theta \implies \frac{dy}{d\theta} = r\cos\theta + \frac{dr}{d\theta}\sin\theta$ **M1A1**

Thus $(y+x)\frac{dy}{dx} = y - x$ becomes $(r\sin\theta + r\cos\theta)\frac{r\cos\theta + \frac{dr}{d\theta}\sin\theta}{-r\sin\theta + \frac{dr}{d\theta}\cos\theta} = r\sin\theta - r\cos\theta$ **M1**

That is $(\sin\theta + \cos\theta)\left(r\cos\theta + \frac{dr}{d\theta}\sin\theta\right) = (\sin\theta - \cos\theta)\left(-r\sin\theta + \frac{dr}{d\theta}\cos\theta\right)$

as $r > 0,\ r \neq 0$

Multiplying out and collecting like terms gives

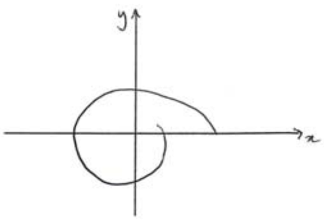
$$r(\cos^2\theta + \sin^2\theta) + \frac{dr}{d\theta}(\sin^2\theta + \cos^2\theta) = 0$$

which is $r + \frac{dr}{d\theta} = 0$. **M1A1* (7)**

So $re^{\theta} + \frac{dr}{d\theta}e^{\theta} = 0$ **M1**

and thus $re^{\theta} = k$, **A1**

$r = ke^{-\theta}$ **A1**



G1 (4)

(or alternatively $\int \frac{1}{r}dr = \int -d\theta$ **M1** so $\ln|r| = -\theta + c$ **A1** and hence $r = ke^{-\theta}$ **A1**)

(ii) $\left(y + x - x(x^2+y^2)\right)\frac{dy}{dx} = y - x - y(x^2+y^2)$

becomes $(r\sin\theta + r\cos\theta - r^3\cos\theta)\frac{r\cos\theta + \frac{dr}{d\theta}\sin\theta}{-r\sin\theta + \frac{dr}{d\theta}\cos\theta} = r\sin\theta - r\cos\theta - r^3\sin\theta$

that is

$(\sin\theta + \cos\theta - r^2\cos\theta)\left(r\cos\theta + \frac{dr}{d\theta}\sin\theta\right) = (\sin\theta - \cos\theta - r^2\sin\theta)\left(-r\sin\theta + \frac{dr}{d\theta}\cos\theta\right)$

Multiplying out and collecting like terms gives

$r\left(\cos^2\theta + \sin^2\theta - r^2(\cos^2\theta + \sin^2\theta)\right) + \frac{dr}{d\theta}(\sin^2\theta + \cos^2\theta) = 0$ **M1**

which is $r - r^3 + \frac{dr}{d\theta} = 0$. **A1**

$\int \frac{1}{r^3 - r} dr = \int d\theta$

$\int \frac{1}{r^3 - r} dr = \int \frac{1}{r(r^2-1)} dr = \int \frac{1}{r(r-1)(r+1)} dr = \int d\theta$ **M1**

So $\int d\theta = \int \frac{1/2}{r-1} + \frac{-1}{r} + \frac{1/2}{r+1} dr$ **A1**

$\theta + k = \frac{1}{2}\ln\left|\frac{(r-1)(r+1)}{r^2}\right|$ **A1**

So

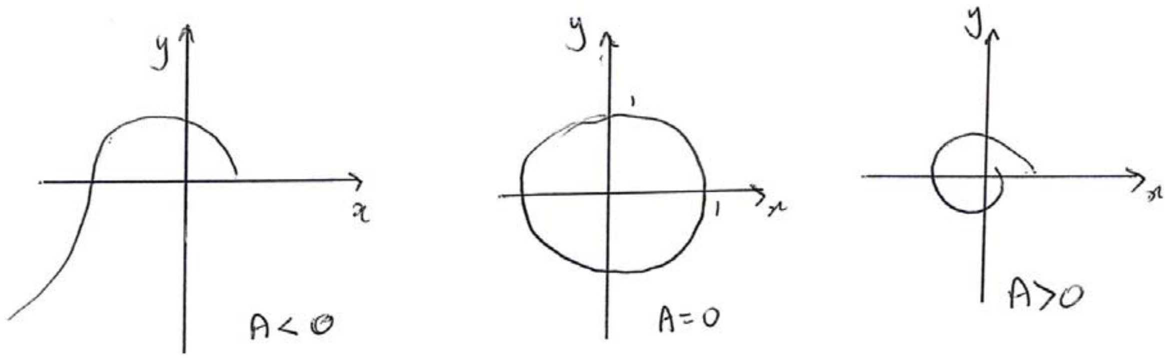
$$\left|\frac{r^2 - 1}{r^2}\right| = Ce^{2\theta}$$

with $C > 0$

$$r^2 = \frac{1}{1 \mp Ce^{2\theta}}$$

that is

$r^2 = \frac{1}{1 + Ae^{2\theta}}$ **A1***



G1 G1 G1 (9)

9. If the initial position of P is , then at time t , $OP^2 = a^2 + x^2$, so conserving energy,

$$\frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2 + \frac{\lambda}{2a}\left(\sqrt{a^2+x^2} - a\right)^2$$

M1 A1 A1

Thus,

$$\dot{x}^2 = v^2 - \frac{\lambda}{ma}\left(\sqrt{a^2+x^2} - a\right)^2$$

M1

i.e.

$$\dot{x}^2 = v^2 - k^2\left(\sqrt{a^2+x^2} - a\right)^2$$

A1* (5)

The greatest value, x_0 , attained by x , occurs when $\dot{x} = 0$. **M1**

Thus $v^2 = k^2\left(\sqrt{a^2+{x_0}^2} - a\right)^2$

So $\sqrt{a^2+{x_0}^2} - a = \frac{v}{k}$ (negative root discounted as all quantities are positive)

Thus

$${x_0}^2 = \left(\frac{v}{k} + a\right)^2 - a^2 = \frac{v^2}{k^2} + \frac{2av}{k}$$

and

$$x_0 = \sqrt{\frac{v^2}{k^2} + \frac{2av}{k}}$$

M1 A1 (3)

As

$$\dot{x}^2 = v^2 - k^2\left(\sqrt{a^2+x^2} - a\right)^2$$

differentiating with respect to t

$$2\dot{x}\ddot{x} = -2k^2\left(\sqrt{a^2+x^2} - a\right)\frac{1}{2}(a^2+x^2)^{\frac{-1}{2}}2x\dot{x}$$

M1 A1

Thus

$$\ddot{x} = -xk^2\frac{\left(\sqrt{a^2+x^2} - a\right)}{\sqrt{a^2+x^2}}$$

A1

So when $x = x_0$, the acceleration of P is

$$-x_0 k^2 \frac{\frac{v}{k}}{\frac{v}{k}+a} = -\sqrt{\frac{v^2}{k^2}+\frac{2av}{k}}\,k^2 \frac{\frac{v}{k}}{\frac{v}{k}+a} = -\frac{kv\sqrt{v^2+2akv}}{v+ak}$$

M1 A1 (5)

$$\dot{x} = \left[v^2 - k^2\left(\sqrt{a^2+x^2}-a\right)^2\right]^{\frac{1}{2}}$$

That is

$$\frac{dx}{dt} = \left[v^2 - k^2\left(\sqrt{a^2+x^2}-a\right)^2\right]^{\frac{1}{2}}$$

and thus

$$\int_0^{\tau/4} dt = \int_0^{x_0} \frac{1}{\left[v^2 - k^2\left(\sqrt{a^2+x^2}-a\right)^2\right]^{\frac{1}{2}}}\,dx$$

where τ is the period. **M1 A1**

So

$$\tau = 4\int_0^{x_0} \frac{1}{\left[v^2 - k^2\left(\sqrt{a^2+x^2}-a\right)^2\right]^{\frac{1}{2}}}\,dx$$

$$\tau = \frac{4}{v}\int_0^{\sqrt{\frac{v^2}{k^2}+\frac{2av}{k}}} \frac{1}{\left[1 - \frac{k^2\left(\sqrt{a^2+x^2}-a\right)^2}{v^2}\right]^{\frac{1}{2}}}\,dx$$

Let

$$u^2 = \frac{k\left(\sqrt{a^2+x^2}-a\right)}{v}$$

B1

then

$$a^2 + x^2 = \left(\frac{vu^2}{k}+a\right)^2$$

and so

$$x^2 = \frac{v^2\,u^4 + 2kavu^2}{k^2}$$

$$x = \sqrt{2kav}\frac{u}{k}\left(1 + \frac{v}{2ka}u^2\right)^{\frac{1}{2}} \approx \sqrt{2kav}\frac{u}{k}$$

as $v \ll ka$

Thus

$$\frac{dx}{du} \approx \frac{1}{k}\sqrt{2kav}$$

M1A1

and so

$$\tau \approx \frac{4}{v}\int_0^1 \frac{1}{\sqrt{1-u^4}}\frac{1}{k}\sqrt{2kav}du = \sqrt{\frac{32a}{kv}}\int_0^1 \frac{1}{\sqrt{1-u^4}}du$$

as required. **M1 A1* (7)**

10. The position vector of the upper particle is

$$\begin{pmatrix} x + a\sin\theta \\ y + a\cos\theta \end{pmatrix}$$

B1 B1

so differentiating with respect to time, its velocity is

$$\begin{pmatrix} \dot{x} + a\,\dot{\theta}\cos\theta \\ \dot{y} - a\,\dot{\theta}\sin\theta \end{pmatrix}$$

E1* (3)

Its acceleration, by differentiating with respect to time, is thus

$$\begin{pmatrix} \ddot{x} + a\,\ddot{\theta}\cos\theta - a\dot{\theta}^2\sin\theta \\ \ddot{y} - a\,\ddot{\theta}\sin\theta - a\dot{\theta}^2\cos\theta \end{pmatrix}$$

M1 A1 A1

so by Newton's second law resolving horizontally and vertically

$$\begin{pmatrix} -T\sin\theta \\ -T\cos\theta - mg \end{pmatrix} = m\begin{pmatrix} \ddot{x} + a\,\ddot{\theta}\cos\theta - a\dot{\theta}^2\sin\theta \\ \ddot{y} - a\,\ddot{\theta}\sin\theta - a\dot{\theta}^2\cos\theta \end{pmatrix}$$

M1 A1

That is

$$m\begin{pmatrix} \ddot{x} + a\,\ddot{\theta}\cos\theta - a\dot{\theta}^2\sin\theta \\ \ddot{y} - a\,\ddot{\theta}\sin\theta - a\dot{\theta}^2\cos\theta \end{pmatrix} = -T\begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix} - mg\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The other particle's equation is

$$m\begin{pmatrix} \ddot{x} - a\,\ddot{\theta}\cos\theta + a\dot{\theta}^2\sin\theta \\ \ddot{y} + a\,\ddot{\theta}\sin\theta + a\dot{\theta}^2\cos\theta \end{pmatrix} = T\begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix} - mg\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

B1 (6)

Adding these two equations we find

$$2m\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = -2mg\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

i.e. $\ddot{x} = 0$ and $\ddot{y} = -g$ **M1 A1***

Thus

$$m\begin{pmatrix} -a\,\ddot{\theta}\cos\theta + a\dot{\theta}^2\sin\theta \\ a\,\ddot{\theta}\sin\theta + a\dot{\theta}^2\cos\theta \end{pmatrix} = T\begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}$$

i.e. $m\left(-a\,\ddot{\theta}\cos\theta + a\dot{\theta}^2\sin\theta\right) = T\sin\theta$ and $m\left(a\,\ddot{\theta}\sin\theta + a\dot{\theta}^2\cos\theta\right) = T\cos\theta$

Multiplying the second of these by $\sin\theta$ and the first by $\cos\theta$ and subtracting,

$ma\ddot{\theta} = 0$ and so $\ddot{\theta} = 0$. **M1A1* (4)**

Thus $\dot{\theta} = a\ constant$ and as initially $2a\dot{\theta} = u$, $\dot{\theta} = \frac{u}{2a}$ **M1 A1**

Therefore the time to rotate by $\frac{1}{2}\pi$ is given by $\tau\dot{\theta} = \frac{1}{2}\pi$, so $\tau = \frac{1}{2}\pi \div \frac{u}{2a} = \frac{\pi a}{u}$ **A1**

As $\ddot{y} = -g$ and initially $\dot{y} = v$, at time , $\dot{y} = v - gt$, and so $y = vt - \frac{1}{2}gt^2 + h$ as the centre of the rod is initially h above the table. **M1 A1**

Hence, given the condition that the particles hit the table simultaneously,

$$0 = v\,\pi a/u - 1/2\; g(\pi a/u)^2 + h$$

.

Hence $0 = 2\pi uva - \pi^2 a^2 g + 2hu^2$, or $2hu^2 = \pi^2 a^2 g - 2\pi uva$ as required. **M1 A1* (7)**

11. (i) Suppose that the force exerted by P on the rod has components X perpendicular to the rod and Y parallel to the rod. Then taking moments for the rod about the hinge, $Xd = 0$, **M1**

which as $d \neq 0$ yields $X = 0$ and hence the force exerted on the rod by P is parallel to the rod. **A1* (2)**

Resolving perpendicular to the rod for P, $mg \sin\alpha = m(r - d\sin\alpha)\omega^2 \cos\alpha$ **M1 A1**

Dividing by $m\omega^2 \sin\alpha$, $\frac{g}{\omega^2} = (r - d\sin\alpha)\cot\alpha$

That is $a = r\cot\alpha - d\cos\alpha$ or in other words $r\cot\alpha = a + d\cos\alpha$ as required. **M1 A1* (4)**

The force exerted by the hinge on the rod is along the rod towards P, **B1**

and if that force is F, then resolving vertically for P, $F\cos\alpha = mg$ **M1 A1**

so $F = mg\sec\alpha$. **A1 (4)**

(ii) Suppose that the force exerted by m_1 on the rod has component X_1 perpendicular to the rod towards the axis, that the force exerted by m_2 on the rod has component X_2 perpendicular to the rod towards the axis, **B1**

then resolving perpendicular to the rod for m_1, $m_1 g \sin\beta + X_1 = m_1(r - d_1 \sin\beta)\omega^2 \cos\beta$ **M1A1**

and similarly for m_2, $m_2 g \sin\beta + X_2 = m_2(r - d_2 \sin\beta)\omega^2 \cos\beta$

M1A1

Taking moments for the rod about the hinge, $X_1 d_1 + X_2 d_2 = 0$ **M1A1**

So multiplying the first equation by d_1, the second by d_2 and adding we have

$$m_1 g\, d_1 \sin\beta + m_2 d_2 g \sin\beta = m_1 d_1 (r - d_1 \sin\beta)\omega^2 \cos\beta + m_2 d_2 (r - d_2 \sin\beta)\omega^2 \cos\beta$$

Dividing by $(m_1 d_1 + m_2 d_2)\omega^2 \sin\beta$, $\frac{g}{\omega^2} = r\cot\beta - \left(\frac{m_1 d_1{}^2 + m_2 d_2{}^2}{m_1 d_1 + m_2 d_2}\right)\cos\beta$ **M1A1**

That is $r\cot\beta = a + b\cos\beta$, where $b = \frac{m_1 d_1{}^2 + m_2 d_2{}^2}{m_1 d_1 + m_2 d_2}$ **A1 (10)**

12. (i) The probability distribution function of S_1 is

S_1	1	2	3	4	5	6
p	$^1/_6$	$^1/_6$	$^1/_6$	$^1/_6$	$^1/_6$	$^1/_6$

so the probability distribution function of R_1 is

R_1	0	1	2	3	4	5
p	$^1/_6$	$^1/_6$	$^1/_6$	$^1/_6$	$^1/_6$	$^1/_6$

and thus $G(x) = \frac{1}{6}(1 + t + t^2 + t^3 + t^4 + t^5)$. **B1**

The probability distribution function of S_2 is

S_2	2	3	4	5	6	7	8	9	10	11	12
p	$^1/_{36}$	$^2/_{36}$	$^3/_{36}$	$^4/_{36}$	$^5/_{36}$	$^6/_{36}$	$^5/_{36}$	$^4/_{36}$	$^3/_{36}$	$^2/_{36}$	$^1/_{36}$

M1

so the probability distribution function of R_2 is

R_2	0	1	2	3	4	5
p	$^6/_{36}$	$^6/_{36}$	$^6/_{36}$	$^6/_{36}$	$^6/_{36}$	$^6/_{36}$

A1

which is the same as for R_1 and hence its probability generating function is also $G(x)$. **A1***

Therefore, the probability generating function of R_n is also $G(x)$ **B1**

and thus the probability that S_n is divisible by 6 is $^1/_6$. **B1 (6)**

(ii) The probability distribution function of T_1 is

T_1	0	1	2	3	4
p	$^1/_6$	$^2/_6$	$^1/_6$	$^1/_6$	$^1/_6$

and thus $G_1(x) = \frac{1}{6}(1 + 2x + x^2 + x^3 + x^4)$ **M1 A1**

$G_2(x)$ would be $\left(G_1(x)\right)^2$ except that the powers must be multiplied congruent to modulus 5.

$G_1(x) = \frac{1}{6}(1 + 2x + x^2 + x^3 + x^4) = \frac{1}{6}(x + 1 + x + x^2 + x^3 + x^4) = \frac{1}{6}(x + y)$ **B1**

Thus $G_2(x)$ would be $\frac{1}{36}(x + y)^2$

except $xy = x(1 + x + x^2 + x^3 + x^4) = x + x^2 + x^3 + x^4 + 1 = y$ **M1A1**

and $y^2 = (1 + x + x^2 + x^3 + x^4)(1 + x + x^2 + x^3 + x^4) = (1 + x + x^2 + x^3 + x^4) + (x + x^2 + x^3 + x^4 + 1) + (x^2 + x^3 + x^4 + 1 + x) + (x^3 + x^4 + 1 + x + x^2) + (x^4 + 1 + x + x^2 + x^3) = 5y$ **A1**

So $G_2(x) = \frac{1}{36}(x + y)^2 = \frac{1}{36}(x^2 + 2xy + y^2) = \frac{1}{36}(x^2 + 2y + 5y) = \frac{1}{36}(x^2 + 7y)$ **M1A1* (8)**

$$G_3(x) = \frac{1}{6^3}(x + y)^3 = \frac{1}{6^3}(x + y)(x^2 + 7y) = \frac{1}{6^3}(x^3 + yx^2 + 7xy + 7y^2)$$

That is

$$G_3(x) = \frac{1}{6^3}(x^3 + yx^2 + 7xy + 7y^2) = \frac{1}{6^3}(x^3 + y + 7y + 35y) = \frac{1}{6^3}(x^3 + 43y)$$

We notice that the coefficient of y inside the bracket in $G_n(x)$ is $(1 + 6 + 6^2 + \cdots 6^{n-1})$

This can be shown simply by induction. It is true for $n = 1$ trivially.

Consider $(x + y)\left(x^r + \left(1 + 6 + 6^2 + \cdots + 6^{k-1}\right)y\right) = x^{r+1} + yx^r + \left(1 + 6 + 6^2 + \cdots + 6^{k-1}\right)xy + \left(1 + 6 + 6^2 + \cdots + 6^{k-1}\right)y^2$

$$yx^r + \left(1 + 6 + 6^2 + \cdots 6^{k-1}\right)xy + \left(1 + 6 + 6^2 + \cdots + 6^{k-1}\right)y^2$$
$$= y + \left(1 + 6 + 6^2 + \cdots + 6^{k-1}\right)y + 5\left(1 + 6 + 6^2 + \cdots + 6^{k-1}\right)y$$

$$5\left(1 + 6 + 6^2 + \cdots 6^{k-1}\right) = (6 - 1)\left(1 + 6 + 6^2 + \cdots 6^{k-1}\right) = 6^k - 1$$

So $y + \left(1 + 6 + 6^2 + \cdots + 6^{k-1}\right)y + 5\left(1 + 6 + 6^2 + \cdots + 6^{k-1}\right)y = \left(1 + 6 + 6^2 + \cdots 6^k\right)y$

as required. **M1**

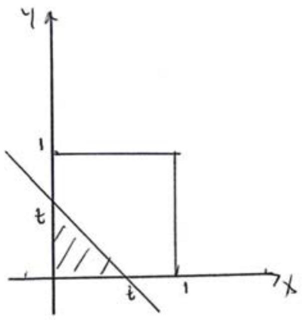
However, this coefficient is the sum of a GP and so $G_n(x) = \frac{1}{6^n}\left(x^{n-5p} + \frac{6^n - 1}{5}y\right)$ where p is an integer such that $0 \leq n - 5p \leq 4$. **M1 A1**

So if n is not divisible by 5, the probability that S_n is divisible by 5 will be the coefficient of x^0 which in turn is the coefficient of y , namely $\frac{1}{6^n}\left(\frac{6^n - 1}{5}\right) = \frac{1}{5}\left(1 - \frac{1}{6^n}\right)$ as required. **B1***

If n is divisible by 5, the probability that S_n is divisible by 5 will be $\frac{1}{6^n}\left(1 + \frac{6^n - 1}{5}\right)$ as $x^{n-5p} = x^0$

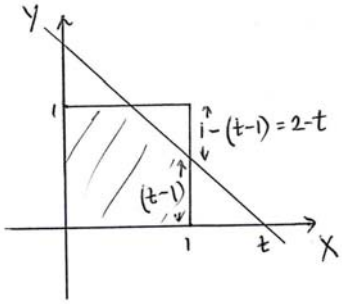
That is $\frac{1}{5}\left(1+\frac{4}{6^n}\right)$ **M1A1 (6)**

13. (i)



G1

$P(X+Y<t) = \frac{1}{2}t^2$ if $0 \le t \le 1$ **B1**



G1

and $P(X+Y<t) = 1-\frac{1}{2}(2-t)^2$ if $1 < t \le 2$ **B1**

$P(X+Y<t) = 0$ if $t<0$ and $P(X+Y<t) = 1$ if $t>2$

So $F(t) = \begin{cases} 0 \ \ for\ t<0 \\ \frac{1}{2}t^2 \ \ for\ 0 \le t \le 1 \\ 1-\frac{1}{2}(2-t)^2 \ \ for\ 1<t\le 2 \\ 1\ for\ t>2 \end{cases}$

B1 (5)

Thus $P((X+Y)^{-1}<t) = P\left(X+Y>\frac{1}{t}\right) = 1-P\left(X+Y<\frac{1}{t}\right)$

$$= \begin{cases} 1-\dfrac{1}{2t^2} \ \ for\ 1 \le t \\ \dfrac{1}{2}\left(2-\dfrac{1}{t}\right)^2 \ \ for\ \dfrac{1}{2} \le t < 1 \\ 0 \ \ for\ t<\dfrac{1}{2} \end{cases}$$

M1 A1

So as $f(t) = \frac{dF(t)}{dt}$,

$$f(t) = \begin{cases} 0 \;\; for\; t < \frac{1}{2} \\ \frac{1}{t^2}\left(2 - \frac{1}{t}\right) \;\; for\; \frac{1}{2} \le t < 1 \\ \frac{1}{t^3} \;\; for\; 1 \le t \end{cases}$$

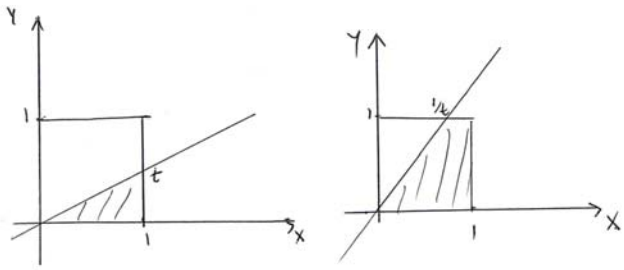
as required. **M1A1* (4)**

$$E\left(\frac{1}{X+Y}\right) = \int_{\frac{1}{2}}^{1} t(2t^{-2} - t^{-3})dt + \int_{1}^{\infty} t.t^{-3}dt = \;[2\ln t + t^{-1}]_{\frac{1}{2}}^{1} + [-t^{-1}]_{1}^{\infty}$$

$$= 1 - 2\ln\frac{1}{2} - 2 + 1 = 2\ln 2$$

M1 A1 (2)

(ii)



G1

$$P\left(\frac{Y}{X} < t\right) = \begin{cases} \frac{1}{2}t \;\; for\; 0 \le t \le 1 \\ 1 - \frac{1}{2}t^{-1} \;\; for\; t > 1 \end{cases}$$

B1 (2)

Thus

$$P\left(\frac{X}{X+Y} < t\right) = P\left(\frac{X+Y}{X} > \frac{1}{t}\right) = P\left(1 + \frac{Y}{X} > \frac{1}{t}\right) = P\left(\frac{Y}{X} > \frac{1}{t} - 1\right) = 1 - P\left(\frac{Y}{X} < \frac{1}{t} - 1\right)$$

So

$$F(t) = \begin{cases} 1 - \frac{1}{2}\left(\frac{1}{t} - 1\right) \;\; for\; \frac{1}{2} \le t \le 1 \\ \frac{1}{2}\left(\frac{1}{t} - 1\right)^{-1} \;\; for\; 0 \le t < \frac{1}{2} \end{cases}$$

i.e.

$$F(t) = \begin{cases} \frac{1}{2}(3 - t^{-1}) \;\; for\; \frac{1}{2} \le t \le 1 \\ \frac{1}{2}\left(\frac{t}{1-t}\right) \;\; for\; 0 \le t < \frac{1}{2} \end{cases}$$

M1A1

So as $f(t)=\frac{dF(t)}{dt}$,

$$f(t)=\begin{cases}\frac{1}{2}t^{-2} & for\ \frac{1}{2}\le t\le 1\\ \frac{1}{2}(1-t)^{-2} & for\ 0\le t<\frac{1}{2}\end{cases}$$

M1A1 (4)

$E\left(\frac{X}{X+Y}\right)=\frac{1}{2}$ because, by symmetry, $E\left(\frac{X}{X+Y}\right)=E\left(\frac{Y}{X+Y}\right)$

and $E\left(\frac{X}{X+Y}\right)+E\left(\frac{Y}{X+Y}\right)=E\left(\frac{X+Y}{X+Y}\right)=E(1)=1$ **B1**

$$E\left(\frac{X}{X+Y}\right)=\int_0^{\frac{1}{2}}\frac{1}{2}t(1-t)^{-2}\,dt+\int_{\frac{1}{2}}^{1}t\times\frac{1}{2}t^{-2}\,dt$$

$$=\left[\frac{1}{2}t(1-t)^{-1}\right]_0^{\frac{1}{2}}-\int_0^{\frac{1}{2}}\frac{1}{2}(1-t)^{-1}dt+\left[\frac{1}{2}\ln t\right]_{\frac{1}{2}}^{1}$$

$$=\frac{1}{2}-\left[-\frac{1}{2}\ln(1-t)\right]_0^{\frac{1}{2}}-\frac{1}{2}\ln\frac{1}{2}$$

$$=\frac{1}{2}+\frac{1}{2}\ln\frac{1}{2}-\frac{1}{2}\ln\frac{1}{2}=\frac{1}{2}$$

as required. **M1A1 (3)**